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On nonanticommutative $\mathcal{N} = 2$ sigma-models in two dimensions

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Abstract: We study nonanticommutative deformations of $\mathcal{N} = 2$ two-dimensional euclidean sigma models. We find that these theories are described by simple deformations of Zumino's lagrangian and the holomorphic superpotential. Geometrically, this deformation can be interpreted as a fuzziness in target space controlled by the vacuum expectation value of the auxiliary field. In the case of nonanticommutative deformations preserving euclidean invariance, we find that a continuation of the deformed supersymmetry algebra to lorentzian signature leads to a rather intriguing central extension of the ordinary (2,2) superalgebra.

Keywords: Field Theories in Lower Dimensions,Sigma Models,Superspaces,Non-Commutative Geometry.

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1. Introduction

The study of deformations of quantum field theories has been the subject of renewed interest since the realization of its relevance in string theory. A large part of these work has been devoted to noncommutative theories where the commutator of the coordinates gets deformed by a central term. Quantum fields defined on noncommutative spaces present a plethora of unexpected properties [1].

More recently it has been realized that a new class of theories with reduced supersymmetry can be constructed by deforming the anticommutator of the fermionic coordinates in superspace. In ref. [2] $\mathcal{N}=1$ supersymmetric theories in four dimensions were deformed by

$$\{\theta^a, \theta^b\} = C^{ab}\,, \qquad C^{ab} \in \mathbb{C}\,, \tag{1.1}$$

while all other anticommutators are equal to zero. On general grounds, a Clifford deformation [3] of the Grassmann algebra of fermionic coordinates can only be consistently carried out in euclidean signature where $\theta^\pm$ and $\overline{\theta}^\pm$ are independent [4]. Eq. (1.1) induces a deformation in the $\mathcal{N}=1$ supersymmetry algebra such that only one half of the original supercharges remain a symmetry of the deformed theory [2, 5 − 7]. This kind of theories emerge also as an effective low energy description of certain superstring theories in constant graviphoton backgrounds [8]. In this set-up the euclidean signature is forced by the requirement that the background field has vanishing back reaction on the metric.

In the case of the nonanticommutative versions of $\mathcal{N} = 1$ super-Yang-Mills theories and the Wess-Zumino model the deformation results in the addition of a finite number of higher dimensional operators to the undeformed lagrangian [2]. This implies in particular that the deformed theory preserves locality, unlike the case of noncommutative field theories where the lagrangian contains and infinite number of terms with an arbitrary high number of derivatives. This type of theories are surprisingly renormalizable in spite of the presence of nominally higher dimensional terms in the lagrangian [9, 10]. In the case of theories with more supersymmetries the situation is richer. For example, nonanticommutative singlet deformations of four-dimensional $\mathcal{N} = 2$ theories [11] give rise to a non-polynomial deformed superpotential [12]. Instantons in nonanticommutative theories have been also a subject of interest [13].

Two-dimensional supersymmetric sigma models with extended supersymmetry are a particularly interesting class of theories to study nonanticommutative deformations. In [14, 15] the deformation of (2,2) two-dimensional sigma-models was considered and it was found that the deformed lagrangian can be written as an infinite series in $\det C$ (see [16] for its four-dimensional analog). Each term, however, contains at most two derivatives, so one may ask whether there is a possibility of rewriting a lagrangian containing a finite number of terms with deformed (field-dependent) couplings.

In this paper we have a closer look at the construction of such nonanticommutative $\mathcal{N} = 2$ sigma-models in two dimensions. In particular we consider a type of nonanticommutative deformation that preserve two-dimensional euclidean invariance. We find that, in this particular case, the supersymmetry algebra is deformed by a central extension depending on the Casimir operator P_+P_-.

Despite the low number of dimensions we work in, and the fact that chirality and charge conjugation are compatible in two-dimensions, it seems not possible to find a "two-dimensional loophole" of [8] (specially the last reference) which would have allowed us to work with lorentzian world-sheets. We can, nevertheless, press on and consider the deformed, centrally extended supersymmetry algebra in Minkowski space-time. In this case we have a central extension of the two-dimensional $\mathcal{N} = 2$ supersymmetry algebra with a central extension compatible with Haag-Łopuszański-Sohnius theorem [17], where the central charges depend on kinematical invariants like the mass of the state. The interesting part is that the superalgebra implies a bound for the masses of the states in the theory. We are not aware of any supersymmetric field theory realizing such an algebra, but its existence would be quite interesting since it automatically includes an ultraviolet cutoff.

We also show that the infinite series found in [14, 15] can be resummed in a compact expression which can be written as a the standard Zumino's lagrangian [18] with deformed Kähler potential and superpotential, plus a finite number of higher dimensional terms with deformed (field-dependent) couplings. Interestingly, the deformation of the Kähler potential $K(z, \overline{z})$ and the holomorphic superpotential $\mathcal{W}(z)$ has the physical interpretation of a smearing in the holomorphic coordinates of the target manifold φ^i controlled by the auxiliary field F^i. Although we study the case of a particular euclidean-invariant deformation $C^{\pm\pm}=0$, $C^{\pm\mp}=1/M$, the fact that the deformed lagrangian depends on C^{ab} only through $\det C$ [14, 15] implies that our results remain valid for other deformation matrices.

The plan of the paper is as follows: in section 2 we briefly review the Weyl map formalism for nonanticommutative theories. Section 3 is devoted to summarizing some general aspects of the deformation of two-dimensional (2,2) supersymmetric theories. Using the formalism of section 2 we calculate in section 3 the deformed superpotential and Kähler potential, and discuss the physical interpretation of the result. Section 4 deals with the discussion of the classical structure of vacua for the deformed theory. Finally in section 5 we summarize our conclusions.

To make the paper more readable some technical details have been postponed to the appendices. In appendix A aspects of Fourier transforms of functions of anticommuting variables are reviewed. Appendix B details some of the calculation of section 3. In appendix C we include a discussion on the representation of the deformed supersymmetry algebra.

2. The Weyl map

Noncommutative field theories can be conveniently formulated using the Weyl map which defines a homomorphism between the noncommutative algebra of functions and an algebra of operators. To begin with let us consider a superspace $\mathbb{R}^{m|2n}$ with anticommuting coordinates $\theta^1,\ldots,\theta^{2n}$. The deformation (3.1) amounts then to deforming the $\mathbb{Z}_2$-graded algebra of functions defined on it. Following the bosonic case the idea is to find a map between this deformed graded algebra of functions and a graded algebra of operators. Consequently, a set of degree one operators $\widehat{Q}^a$ $(a,b=1,\ldots,2n)$ is then introduced satisfying the deformed anticommutation relations

$$\{\widehat{Q}^a,\widehat{Q}^b\}=C^{ab}\,. \tag{2.1}$$

In terms of them any superspace function $f(\theta)$ has associated its Weyl transform (or Weyl symbol) through the definition

$$\widehat{f}\equiv(-1)^n\int d^{2n}\eta\, e^{-\eta_a\widehat{Q}^a}\,\widetilde{f}(\eta)\,, \tag{2.2}$$

where $\widetilde{f}(\eta)$ is the Fourier transform of $f(\theta)$ (see appendix A for notation and definitions). This definition of the symbol associated to a superspace function $f(\theta)$ will be very important in the calculation of the action of the deformed lagrangian in the next section.

The set of Weyl symbols form another $\mathbb{Z}_2$-graded algebra with $\deg(\widehat{f})=\deg(f)$. Actually, the map (2.2) defines a homomorphism between the two algebras in which the product of two symbols $\widehat{f}$, $\widehat{g}$ is associated with the noncommutative product of the corresponding functions

$$\widehat{f}\,\widehat{g}=\int d^{2n}\eta\, e^{-\eta_a\widehat{Q}^a}\,\widetilde{(f\star g)}(\eta)\,, \tag{2.3}$$

where the star-product is defined by[1]

$$f(\theta)\star g(\theta)=f(\theta)\exp\left(-\frac{1}{2}C^{ab}\,\frac{\overleftarrow{\partial}}{\partial\theta^{a}}\frac{\overrightarrow{\partial}}{\partial\theta^{b}}\right)g(\theta)\,. \tag{2.5}$$

The function-symbol map (2.2) can be also conveniently expressed using an operator $\widehat{\Xi}(\theta)$ defined by

$$\widehat{\Xi}(\theta)=(-1)^{n}\int d^{2n}\eta\, e^{-\eta_{a}(\widehat{Q}^{a}-\theta^{a})}\,. \tag{2.6}$$

A trivial manipulation of (2.2) shows that $\widehat{f}$ can be written as the integral

$$\widehat{f}=\int d^{2n}\theta\,\widehat{\Xi}(\theta)\,f(\theta)\,. \tag{2.7}$$

Using eq. (2.6) it is easy to construct the inverse of the Weyl map by defining a supertrace operation on the algebra of symbols with the following properties

$$\begin{aligned}\mathrm{str}\,\mathbf{1}&=0\,,\\ \mathrm{str}\left[\widehat{Q}^{a_1}\cdots\widehat{Q}^{a_k}\right]&=0\,,\qquad k=1,\ldots,2n-1\\ \mathrm{str}\left[\widehat{Q}^{1}\cdots\widehat{Q}^{2n}\right]&=1\,.\end{aligned} \tag{2.8}$$

Then, given a symbol $\widehat{f}$ the function $f(\theta)$ associated with it by the map (2.2) is given by

$$f(\theta)=\mathrm{str}\left[\widehat{f}\,\widehat{\Xi}(\theta)\right]. \tag{2.9}$$

For later use it is interesting to particularize some of the formulae to the case $n=1$ with fermionic coordinates $\theta^{\pm}$. In this case the symbol $\widehat{f}$ associated with the function $f(\theta^{\pm})$ is defined in terms of its Fourier transform $\widetilde{f}(\eta_{\pm})$ as

$$\widehat{f}=-\int d^{2}\eta\, e^{-(\eta_{+}\widehat{Q}^{+}+\eta_{-}\widehat{Q}^{-})}\,\widetilde{f}(\eta_{\pm})\,, \tag{2.10}$$

whereas the inverse map is constructed as in eq. (2.7) with $\widehat{\Xi}(\theta^{\pm})$ given by

$$\widehat{\Xi}(\theta^{\pm})=-\int d^{2}\eta\,\exp\left[\eta_{+}\left(\widehat{Q}^{+}-\theta^{+}\right)+\eta_{-}\left(\widehat{Q}^{-}-\theta^{-}\right)\right]. \tag{2.11}$$

The Weyl formalism gives automatically a prescription for the definition of functions of the superfields defined on the deformed superspace. Given a function $F(\Phi)$ of a superfield $\Phi(\theta)$ the corresponding operator $F(\widehat{\Phi})$ is defined by

$$F(\widehat{\Phi})\equiv\sum_{n=0}^{\infty}\frac{1}{n!}F^{(n)}(0)\widehat{\Phi}^{n}\,. \tag{2.12}$$

[1] We follow the conventions of ref. [2] and define right derivations as

$$f(\theta)\,\frac{\overleftarrow{\partial}}{\partial\theta^{a}}=(-1)^{\deg(f)}\,\frac{\overrightarrow{\partial}}{\partial\theta^{a}}\,f(\theta) \tag{2.4}$$

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The deformed function $F(\Phi)_\star$ is then obtained by applying the inverse Weyl map

$$F(\Phi)_\star = \mathrm{str}\left[\widehat{\Xi}(\theta)\, F(\widehat{\Phi})\right] = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0)\, \Phi(\theta)\star \overset{(n)}{\cdots} \star\Phi(\theta)\,. \tag{2.13}$$

This recovers the usual prescription for the definition of functions of superfields in nonanticommutative superspaces (see, for example, [14, 15]).

3. Nonanticommutative theories in two dimensions

In the following we focus our analysis to the case of two-dimensional (2,2) superspace $\mathbb{R}^{2|4}$ with coordinates $x^\pm$, $\theta^\pm$ and $\overline{\theta}^\pm$. Although the theory is defined in euclidean space we use minkowskian notation throughout and consider the euclidean invariant deformation

$$\{\theta^+, \theta^-\} = \frac{1}{M}\,, \qquad (\theta^\pm)^2 = 0\,, \tag{3.1}$$

with $M \in \mathbb{R}$ the characteristic energy scale of the deformation and all the anticommutators of antiholomorphic coordinates unchanged. Since we are working in euclidean space, it is very important to keep in mind that bars do not denote complex conjugation, $\overline{\theta}^\pm \neq (\theta^\pm)^*$. This theory can be obtained by dimensional reduction from the corresponding $\mathcal{N} = 1/2$ theory in four dimensions [15] by taking $C^{11} = C^{22} = 0$, $C^{12} = C^{21} = 1/M$.

3.1 The deformed superalgebra

In order to make the chiral structure of the theory explicit it is convenient to introduce chiral coordinates $y^\pm$ given by

$$y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm\,, \tag{3.2}$$

which satisfy $[y^+, y^-] = 0$ provided that

$$[x^\pm, \theta^\mp] = -\frac{i}{M}\overline{\theta}^\pm\,, \qquad [x^+, x^-] = \frac{1}{M}\overline{\theta}^+\overline{\theta}^-\,, \tag{3.3}$$

with all other commutators involving $x^\pm$ equal to zero. Conditions (3.3) also imply that $y^\pm$ commute with the holomorphic fermionic coordinates $\theta^\pm$. This fact is actually crucial to apply the formalism of the previous section, since it allows to consider $y^\pm$ as spectator coordinates in the Weyl map.

In terms of $y^\pm$ the supercovariant derivatives are defined in the usual way [2, 15]

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - 2i\overline{\theta}^\pm\frac{\partial}{\partial y^\pm}\,, \qquad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm}\,, \tag{3.4}$$

and similarly for the supercharges $Q_\pm$, $\overline{Q}_\pm$

$$Q_\pm = \frac{\partial}{\partial\theta^\pm}\,, \qquad \overline{Q}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - 2i\theta^\pm\frac{\partial}{\partial y^\pm}\,. \tag{3.5}$$

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The deformation (3.1) implies that the algebra of supercharges and supercovariant derivatives remain unchanged except for the only anticommutator

$$\{\overline{Q}_+, \overline{Q}_-\} = -\frac{4}{M}\frac{\partial^2}{\partial y^+ \partial y^-} \,. \tag{3.6}$$

As a result the original (2,2) supersymmetry is broken down to the supersymmetries generated by $Q_\pm$ [2].

It is rather remarkable that, when expressed in momentum space, the particular deformation of the algebra (3.6) can be written as

$$\{\overline{Q}_+, \overline{Q}_-\} = \frac{4}{M} P_+ P_- \,. \tag{3.7}$$

Actually, P_+P_- is a Casimir operator of the two-dimensional euclidean group and therefore the deformed algebra is a central extension of the (2,2) superalgebra. It is important to stress that this central extension (3.7) only arises for the deformation (3.1) that preserves euclidean invariance. Similarly, in dimensions higher than two any non-vanishing C^{ab} breaks the full euclidean invariance of the theory. As a consequence the deformation of the supersymmetry algebra is not a central extension, unlike the case studied here.

Although we are forced to work in euclidean space, it is interesting to look at the central extension (3.7) from a minkowskian point of view. In principle, such a central extension of the two-dimensional (2,2) supersymmetry algebra is allowed by the Haag-Łopuszański-Sohnius theorem [17]. Interestingly, following the arguments of ref. [19] shows that such a central extension implies an upper bound in the spectrum of eigenvalues of the mass of the states, given by the Casimir operator P_+P_-. This bound is determined by the energy scale of the deformation M,

$$P_+P_- \leq \frac{M^2}{4} \,. \tag{3.8}$$

In appendix C we have given a detailed calculation leading to this result. It is still to be seen, however, whether this deformed algebra can be somehow realized in a Quantum Field Theory in Minkowski space-time.

3.2 Chiral and antichiral superfields

To fix notation, in the following we will outline the construction of chiral and antichiral superfields done in refs. [2, 14, 15]. Because of the nonanticommutativity of the fermionic coordinates $\theta^\pm$, an ordering prescription is required in the definition and multiplication of superfields. Weyl (symmetric) ordering is implemented by introducing the nonanticommutative star-product of eq. (2.5) [2], which for the particular deformation (3.1) reads

$$\begin{aligned} f(\theta^\pm) \star g(\theta^\pm) &= f(\theta^\pm) \exp\left[-\frac{1}{2M}\left(\frac{\overleftarrow{\partial}}{\partial\theta^+}\frac{\overrightarrow{\partial}}{\partial\theta^-} + \frac{\overleftarrow{\partial}}{\partial\theta^-}\frac{\overrightarrow{\partial}}{\partial\theta^+}\right)\right] g(\theta^\pm) \\ &= fg - \frac{1}{2M}\left(\frac{\partial f}{\partial\theta^+}\frac{\partial g}{\partial\theta^-} + \frac{\partial f}{\partial\theta^-}\frac{\partial g}{\partial\theta^+}\right) + \frac{1}{4M^2}\left(\frac{\partial^2 f}{\partial\theta^+\partial\theta^-}\right)\left(\frac{\partial^2 g}{\partial\theta^+\partial\theta^-}\right), \end{aligned} \tag{3.9}$$

for any two functions $f(\theta^\pm)$, $g(\theta^\pm)$ of degree zero. The difference between the star-product and the ordinary product is a total derivative in $\theta^\pm$.

Given that $D_\pm$, $\overline{D}_\pm$ anticommute with one another, chiral and antichiral superfields can be defined in the usual way by[2]

$$\overline{D}_\pm \Phi = 0 \quad (\text{chiral})\,, \qquad D_\pm \overline{\Phi} = 0 \quad (\text{antichiral})\,. \tag{3.10}$$

The constraint for chiral superfields can be easily solved in terms of $y^\pm$ as

$$\Phi(y^\pm, \theta^\pm) = \varphi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm)\,. \tag{3.11}$$

This expression is automatically Weyl ordered, since $\theta^+\theta^- = \frac{1}{2}(\theta^+\theta^- - \theta^-\theta^+)$.

Similarly, antichiral superfields $\overline{\Phi}$ are functions only of $\overline{\theta}^\pm$ and the antichiral coordinates

$$\overline{y}^\pm = y^\pm + 2i\theta^\pm\overline{\theta}^\pm\,. \tag{3.12}$$

Using eq. (3.3) one sees that the $\overline{y}^\pm$ do not commute among themselves or with $\theta^\pm$. However the component fields can be expanded around the commuting coordinates $y^\pm$. As we will see below, it is convenient to write the antichiral superfield $\overline{\Phi}$ as a (0,2) superfield whose component fields are themselves (2,0) superfields [2, 14, 15]:

$$\begin{aligned}
\overline{\Phi}(\overline{y}^\pm, \overline{\theta}^\pm) &= \overline{\varphi}(\overline{y}^\pm) + \overline{\theta}^+\overline{\psi}_+(\overline{y}^\pm) + \overline{\theta}^-\overline{\psi}_-(\overline{y}^\pm) + \overline{\theta}^+\overline{\theta}^- F(\overline{y}^\pm) \qquad (3.13)\\
&= \overline{\varphi}(y^\pm) + \overline{\theta}^+\Big[\overline{\psi}_+(y^\pm) - 2i\theta^+\partial_+\overline{\varphi}(y^\pm)\Big] + \overline{\theta}^-\Big[\overline{\psi}_-(y^\pm) - 2i\theta^-\partial_-\overline{\varphi}(y^\pm)\Big] + \\
&\quad + \overline{\theta}^+\overline{\theta}^-\Big[\overline{F}(y^\pm) + 2i\theta^+\partial_+\overline{\psi}_-(y^\pm) - 2i\theta^-\partial_-\overline{\psi}_+(y^\pm) + 4\theta^+\theta^-\partial_+\partial_-\overline{\varphi}(y^\pm)\Big]\,.
\end{aligned}$$

Here and in the following $\partial_\pm$ indicates derivatives with respect to $y^\pm$. To simplify the notation, from now on we will not indicate explicitly the dependence of the superfields on the chiral coordinates $y^\pm$ whenever there is no risk of ambiguity.

The unbroken supersymmetries generated by $Q_\pm$ act by shifting the holomorphic fermionic coordinates, $\theta^\pm \to \theta^\pm + \varepsilon^\pm$ at fixed $y^\pm$ and $\overline{\theta}^\pm$. The transformation of the component of a chiral superfield under $\varepsilon^+Q_+ + \varepsilon^-Q_-$ are given then by

$$\begin{aligned}
\delta\varphi &= \varepsilon^+\psi_+ + \varepsilon^-\psi_-\,,\\
\delta\psi_+ &= \varepsilon^- F\,,\\
\delta\psi_- &= -\varepsilon^+ F\,,\\
\delta F &= 0\,,
\end{aligned} \tag{3.14}$$

while for the components of the antichiral multiplet one finds

$$\begin{aligned}
\delta\overline{\varphi} &= 0\,,\\
\delta\overline{\psi}_+ &= -2i\varepsilon^+\partial_+\overline{\varphi}\,,\\
\delta\overline{\psi}_- &= -2i\varepsilon^-\partial_-\overline{\varphi}\,,\\
\delta\overline{F} &= 2i\varepsilon^+\partial_+\overline{\psi}_- - 2i\varepsilon^-\partial_-\overline{\psi}_+\,.
\end{aligned} \tag{3.15}$$

[2]Twisted chiral or antichiral superfields can also be defined [15]. Here, however, we only deal with chiral and antichiral fields.

4. Two-dimensional Kähler sigma-models

In [14, 15] an explicit construction of the nonanticommutative deformation of two-dimensional supersymmetric theories was given. The resulting lagrangian can be written as an infinite series in $(\det C)^{\frac{1}{2}} F^i$, where F^i is the highest component of the chiral superfield. In the following we will show how this series can be resummed giving rise to expressions making the physical interpretation more accessible.

The subject of our study here is the nonanticommutative deformation (3.1) of the (2,2) Kähler sigma-model in euclidean space described by the lagrangian

$$\mathcal{L} = \int d^2\theta\, d^2\overline{\theta}\, K(\Phi^i, \overline{\Phi}^{\overline{\imath}}) + \int d^2\theta\, \mathcal{W}(\Phi^i) + \int d^2\overline{\theta}\, \mathcal{W}(\overline{\Phi}^{\overline{\imath}})\,, \tag{4.1}$$

where Φ^i, $\overline{\Phi}^{\overline{\imath}}$ $(i, \overline{\imath} = 1, \ldots, N)$ are respectively a set of chiral and antichiral superfields. We remind the reader once more that, since we work in euclidean space, overlines should not be interpreted as complex conjugation. In order to construct the deformed lagrangian we use the prescription given in section 2 to define functions of the superfields.

4.1 The superpotential

We start with the holomorphic superpotential. Beginning with the following function of the Weyl symbol $\widehat{\Phi}$

$$\mathcal{W}(\widehat{\Phi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{i_1} \cdots \partial_{i_n} \mathcal{W}(0) \widehat{\Phi}^{i_1} \cdots \widehat{\Phi}^{i_n}\,, \tag{4.2}$$

and applying the inverse Weyl map we can compute the deformed superpotential. Therefore our first task is to compute the monomial

$$\begin{aligned} \widehat{\Phi}^{i_1} \cdots \widehat{\Phi}^{i_n} &= (-1)^n \int d^2\eta_1 \cdots \int d^2\eta_n\, \widetilde{\Phi}^{i_1}(\eta_1) \cdots \widetilde{\Phi}^{i_n}(\eta_n) \exp\left[-\left(\sum_{k=1}^{n} \eta_{k,a}\right) \widehat{Q}^a\right] \times \\ &\times \exp\left[\frac{1}{2} \sum_{k<j}^{n} \langle \eta_k, \eta_j \rangle\right]\,, \end{aligned} \tag{4.3}$$

where we have defined

$$\langle \eta_i, \eta_j \rangle \equiv C^{ab} \eta_{i,a} \eta_{j,b} = \frac{1}{M} (\eta_{i,+}\eta_{j,-} + \eta_{i,-}\eta_{j,+})\,. \tag{4.4}$$

Replacing η_n by a the new coordinate $\zeta_a = \sum_{i=1}^{n} \eta_{i,a}$, eq. (4.3) can be rewritten as

$$\begin{aligned} \widehat{\Phi}^{i_1} \cdots \widehat{\Phi}^{i_n} &= (-1)^n \int d^2\zeta\, e^{-\zeta_a \widehat{Q}^a} \int d^2\eta_1 \cdots \int d^2\eta_{n-1}\, \widetilde{\Phi}^{i_1}(\eta_1) \cdots \widetilde{\Phi}^{i_n}\left(\zeta - \sum_{i=1}^{n-1} \eta_i\right) \times \\ &\times \exp\left[\frac{1}{2} \sum_{k<j}^{n-1} \langle \eta_k, \eta_j \rangle\right] \exp\left[-\frac{1}{2} \sum_{i=1}^{n-1} \langle \zeta, \eta_i \rangle\right]. \end{aligned} \tag{4.5}$$

Finally, from this expression and using (2.3) we get

$$(\Phi^{i_1} \widetilde{\star \cdots \star} \Phi^{i_n})(\zeta) = (-1)^{n-1} \int d^2\eta_1 \cdots \int d^2\eta_{n-1}\, \widetilde{\Phi}^{i_1}(\eta_1) \cdots \widetilde{\Phi}^{i_n}\left(\zeta - \sum_{i=1}^{n-1} \eta_i\right) \times$$
$$\times \exp\left[\frac{1}{2}\sum_{k<j}^{n-1} \langle \eta_k, \eta_j \rangle\right] \exp\left[-\frac{1}{2}\sum_{i=1}^{n-1} \langle \zeta, \eta_i \rangle\right]. \tag{4.6}$$

In order to compute the contribution of the holomorphic superpotential to the lagrangian we need to keep only the highest component of $\Phi^{i_1}(\theta) \star \cdots \star \Phi^{i_n}(\theta)$. Using the identity (A.9) this is just given by

$$\begin{aligned}
\Phi^{i_1}(\theta) \star \cdots \star \Phi^{i_n}(\theta)\Big|_{\theta^+\theta^-} &= (\Phi^{i_1} \widetilde{\star \cdots \star} \Phi^{i_n})(\zeta)\Big|_{\zeta_\pm = 0} \\
&= (-1)^{n-1} \int d^2\eta_1 \cdots \int d^2\eta_{n-1}\, \widetilde{\Phi}^{i_1}(\eta_1) \cdots \widetilde{\Phi}^{i_n}\left(-\sum_{i=1}^{n-1} \eta_i\right) \times \\
&\quad \times \exp\left[\frac{1}{2}\sum_{k<j}^{n-1} \langle \eta_k, \eta_j \rangle\right],
\end{aligned} \tag{4.7}$$

where the Fourier transforms $\widetilde{\Phi}^i(\eta)$ are expressed in terms of the components of the chiral superfield $\Phi^i(\theta)$ by (see appendix A)

$$\widetilde{\Phi}^i(\eta) = F^i + \eta_+ \psi^i_- - \eta_- \psi^i_+ - \eta_+ \eta_- \varphi^i\,. \tag{4.8}$$

The calculation of the integral in eq. (4.7) is lengthy but straightforward. In particular, from eq. (4.2) we see that only the part symmetric in the indices $\{i_1, \ldots, i_n\}$ contributes. Keeping this in mind we find the result (see appendix B for details)

$$\begin{aligned}
\Phi^{i_1}(\theta) \star \cdots \star \Phi^{i_n}(\theta)\Big|_{\theta^+\theta^-} &= \sum_{k=1}^{n} F^{i_k} \frac{\partial}{\partial \varphi^{i_k}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi \left(\varphi^{i_1} + \frac{\xi}{M} F^{i_1}\right) \cdots \left(\varphi^{i_n} + \frac{\xi}{M} F^{i_n}\right) \\
&\quad - \sum_{k<\ell}^{n} \left(\psi^{i_k}_+ \psi^{i_\ell}_- - \psi^{i_k}_- \psi^{i_\ell}_+\right) \frac{\partial^2}{\partial \varphi^{i_k} \partial \varphi^{i_\ell}} \times \\
&\quad \times \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi \left(\varphi^{i_1} + \frac{\xi}{M} F^{i_1}\right) \cdots \left(\varphi^{i_n} + \frac{\xi}{M} F^{i_n}\right).
\end{aligned} \tag{4.9}$$

Plugging this into the series expansion of the superpotential leads to the surprisingly simple result

$$\int d^2\theta\, \mathcal{W}(\Phi) = F^i \partial_i \mathcal{W}_0(\varphi, F) - \psi^i_+ \psi^j_- \partial_i \partial_j \mathcal{W}_0(\varphi, F)\,, \tag{4.10}$$

where we have used the notation

$$\mathcal{W}_0(\varphi, F) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\, \mathcal{W}\left(\varphi^i + \frac{\xi}{M} F^i\right). \tag{4.11}$$

The effect of the deformation on the holomorphic superpotential amounts then to an averaging of the value of $\mathcal{W}$ around φ^i on a scale set by F^i/M. Nonanticommutativity induces then a certain fuzziness controlled by the auxiliary field.

A similar analysis can be carried out for the antiholomorphic superpotential. Since the anticommutation relations of the coordinates $\overline{\theta}^{\pm}$ are not deformed we will perform the Weyl map only with respect to the holomorphic coordinates $\theta^{\pm}$. This means that the symbols $\widehat{\overline{\Phi}}^{\bar\imath}$ associated with the antichiral superfields are themselves (1,1) superfields with respect to the broken supersymmetries $\overline{Q}_{\pm}$

$$\widehat{\overline{\Phi}}^{\bar\imath} = \widehat{\overline{\sigma}}^{\bar\imath} + \overline{\theta}^{+}\widehat{\overline{\lambda}}_{+}^{\bar\imath} + \overline{\theta}^{-}\widehat{\overline{\lambda}}_{-}^{\bar\imath} + \overline{\theta}^{+}\overline{\theta}^{-}\widehat{\overline{K}}^{\bar\imath}, \tag{4.12}$$

where $\widehat{\overline{\sigma}}$, $\widehat{\overline{\lambda}}_{-}$ and $\widehat{\overline{K}}$ are the symbols associated with the corresponding (2,0) superfields inside of eq. (3.13)

$$\begin{aligned}
\overline{\sigma}^{\bar\imath} &= \overline{\varphi}^{\bar\imath}, \\
\overline{\lambda}_{\pm}^{\bar\imath} &= \overline{\psi}_{\pm}^{\bar\imath} - 2i\theta^{\pm}\partial_{\pm}\overline{\varphi}^{\bar\imath}, \\
\overline{K}^{\bar\imath} &= \overline{F}^{\bar\imath} + 2i\theta^{+}\partial_{+}\overline{\psi}_{-}^{\bar\imath} + 2i\theta^{-}\partial_{-}\overline{\psi}_{+}^{\bar\imath} + 4\theta^{+}\theta^{-}\partial_{+}\partial_{-}\overline{\varphi}^{\bar\imath}.
\end{aligned} \tag{4.13}$$

Notice that since $\overline{\sigma}^{\bar\imath}$ is independent of $\theta^{\pm}$ its symbol is just the c-number $\overline{\varphi}^{\bar\imath}$ itself.

The antiholomorphic superpotential is given by the expansion

$$\overline{\mathcal{W}}(\widehat{\overline{\Phi}}) = \sum_{n=0}^{\infty} \frac{1}{n!}\partial_{\bar\imath_1}\cdots\partial_{\bar\imath_n}\overline{\mathcal{W}}(0)\widehat{\overline{\Phi}}^{\bar\imath_1}\cdots\widehat{\overline{\Phi}}^{\bar\imath_n}. \tag{4.14}$$

Here, however, only the part proportional to $\overline{\theta}^{+}\overline{\theta}^{-}$ contributes to the lagrangian. Therefore the only relevant terms in eq. (4.14) are the ones of the form

$$\widehat{\overline{K}}^{\bar\imath_1}\,\widehat{\overline{\sigma}}^{\bar\imath_2}\cdots\widehat{\overline{\sigma}}^{\bar\imath_n}, \qquad \widehat{\overline{\lambda}}_{+}^{\bar\imath_1}\,\widehat{\overline{\lambda}}_{-}^{\bar\imath_2}\,\widehat{\overline{\sigma}}^{\bar\imath_3}\cdots\widehat{\overline{\sigma}}^{\bar\imath_n} \tag{4.15}$$

and permutations. We can proceed then as in the holomorphic case by identifying the corresponding Fourier transforms. Unlike the calculation of the holomorphic part, here we are interested in the lowest component in $\theta^{+}\theta^{-}$. From eq. (A.8) we see that the relevant terms are the symmetric parts of

$$-\Big(\bar{K}^{\bar\imath_1}\star\widetilde{\bar{\sigma}^{\bar\imath_2}\star\cdots\star\bar{\sigma}^{\bar\imath_n}}\Big)(\zeta)\Big|_{\zeta_+\zeta_-}, \qquad -\Big(\bar{\lambda}_{+}^{\bar\imath_1}\star\widetilde{\bar{\lambda}_{-}^{\bar\imath_2}\star\bar{\sigma}^{\bar\imath_3}}\star\cdots\star\bar{\sigma}^{\bar\imath_n}\Big)(\zeta)\Big|_{\zeta_+\zeta_-}.$$

Proceedings in this way we find

$$\begin{aligned}
\overline{\Phi}^{\bar\imath_1}\star\cdots\star\overline{\Phi}^{\bar\imath_n}\Big|_{\theta^+\theta^-,\mathrm{sym}} &= \sum_{k=1}^{n}\overline{F}^{\bar\imath_k}\frac{\partial}{\partial\overline{\varphi}^{\bar\imath_k}}(\overline{\varphi}^{\bar\imath_1}\cdots\overline{\varphi}^{\bar\imath_n}) - \\
&\quad - \sum_{k<\ell}^{n}\left(\overline{\psi}_{+}^{\bar\imath_k}\,\overline{\psi}_{-}^{\bar\imath_\ell} - \overline{\psi}_{-}^{\bar\imath_k}\,\overline{\psi}_{+}^{\bar\imath_\ell}\right)\frac{\partial^2}{\partial\overline{\varphi}^{\bar\imath_k}\partial\overline{\varphi}^{\bar\imath_\ell}}\left(\overline{\varphi}^{\bar\imath_1}\cdots\overline{\varphi}^{\bar\imath_n}\right).
\end{aligned} \tag{4.16}$$

The first interesting thing is that all dependence in the deformation scale disappears. Actually all terms depending on M disappear after symmetrization. Therefore the antiholomorphic superpotential does not suffer any deformation and we retrieve the standard expression

$$\int d^2\overline{\theta}\,\overline{\mathcal{W}}(\overline{\Phi}) = \overline{F}^{\bar\imath}\,\partial_{\bar\imath}\overline{\mathcal{W}}(\overline{\varphi}) - \overline{\psi}_+^{\bar\imath}\,\overline{\psi}_-^{\bar\jmath}\,\partial_{\bar\imath}\,\partial_{\bar\jmath}\,\overline{\mathcal{W}}(\overline{\varphi})\,. \tag{4.17}$$

Equations (4.10) and (4.17) show that the noncommutative deformation only affects the holomorphic part of the superpotential [2]. It is quite remarkable, however, that the particular deformation suffered by the holomorphic superpotential has a clear geometric interpretation as smearing in the target space coordinates.

4.2 The Kähler potential

After the analysis of the superpotential we turn our attention to the deformation of the Kähler potential. Using eqs. (2.2) and (2.3) we construct the corresponding Kähler function $K(\widehat{\Phi},\widehat{\overline{\Phi}})$ for the symbols $\widehat{\Phi}^i$, $\widehat{\overline{\Phi}}^{\bar\jmath}$ as

$$K(\widehat{\Phi},\widehat{\overline{\Phi}}) = \sum_{n,m=0}^{\infty}\frac{1}{(n+m)!}\partial_{i_1}\cdots\partial_{i_n}\partial_{\bar\jmath_1}\cdots\partial_{\bar\jmath_m}K(0,0)\,\widehat{\Phi}^{(i_1}\cdots\widehat{\Phi}^{i_n}\widehat{\overline{\Phi}}^{\bar\jmath_1}\cdots\widehat{\overline{\Phi}}^{\bar\jmath_m)}\,, \tag{4.18}$$

where we have to consider all the possible permutations of $\widehat{\Phi}^i$'s and $\widehat{\overline{\Phi}}^{\bar\imath}$'s. As in the case of our discussion of the antiholomorphic superpotential, the fact that only $\overline{\theta}^+\overline{\theta}^-$ contributes to the lagrangian implies that in computing the different monomials in eq. (4.18) the only terms that we have to take into consideration are the ones of the form

$$\widehat{\Phi}^{i_1}\cdots\widehat{\Phi}^{i_n}\widehat{\overline{K}}^{\bar\imath_1}\,\widehat{\overline{\sigma}}^{\bar\imath_2}\cdots\widehat{\overline{\sigma}}^{\bar\imath_m}\,,\qquad \widehat{\Phi}^{i_1}\cdots\widehat{\Phi}^{i_n}\widehat{\overline{\lambda}}_+^{\bar\imath_1}\,\widehat{\overline{\lambda}}_-^{\bar\imath_2}\,\widehat{\overline{\sigma}}^{\bar\imath_3}\cdots\widehat{\overline{\sigma}}^{\bar\imath_m} \tag{4.19}$$

in all possible orderings. The calculation of the lagrangian can now be carried out using the same techniques used to evaluate the superpotential (see appendix B). First the Fourier transform of the corresponding monomials is evaluated. The term contributing to the lagrangian corresponds then to the lowest component of the Fourier transform, according to eq. (A.9). At the end, the resulting lagrangian can be written

$$\int d^2\theta\,d^2\overline{\theta}\,K(\Phi,\overline{\Phi}) = \mathcal{L}_0 + \mathcal{L}_1\,, \tag{4.20}$$

where

$$\begin{aligned}
\mathcal{L}_0 = {}& 4\,\partial_{\bar\jmath}\,K_0(\varphi,F,\overline{\varphi})\,\partial_+\partial_-\overline{\varphi}^{\bar\jmath} + \partial_i\,\partial_{\bar\jmath}\,K_0(\varphi,F,\overline{\varphi})\left(2i\psi_+^i\,\partial_-\,\overline{\psi}_-^{\bar\jmath} + 2i\psi_-^i\,\partial_+\,\overline{\psi}_-^{\bar\jmath} + F^i\,\overline{F}^{\bar\jmath}\right) + \\
&+4\,\partial_{\bar\jmath}\,\partial_{\bar\ell}\,K_0(\varphi,F,\overline{\varphi})\,\partial_+\overline{\varphi}^{\bar\jmath}\,\partial_-\,\overline{\varphi}^{\bar\ell} - \partial_i\,\partial_k\,\partial_{\bar\jmath}\,K_0(\varphi,F,\overline{\varphi})\,\psi_+^i\,\psi_-^k\,\overline{F}^{\bar\jmath} + \\
&+\partial_i\,\partial_{\bar\jmath}\,\partial_{\bar\ell}\,K_0(\varphi,F,\overline{\varphi})\left(2i\,\psi_+^i\,\overline{\psi}_+^{\bar\jmath}\,\partial_-\overline{\varphi}^{\bar\ell} + 2i\,\psi_-^i\,\overline{\psi}_-^{\bar\jmath}\,\partial_+\overline{\varphi}^{\bar\ell} - F^i\,\overline{\psi}_+^{\bar\jmath}\,\overline{\psi}_-^{\bar\ell}\right) + \\
&+\partial_i\,\partial_k\,\partial_{\bar\jmath}\,\partial_{\bar\ell}\,K_0(\varphi,F,\overline{\varphi})\,\psi_+^i\,\psi_-^k\,\overline{\psi}_+^{\bar\jmath}\,\overline{\psi}_-^{\bar\ell}\,,
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
\mathcal{L}_1 = {}& \frac{4}{M}\partial_i\,\partial_{\bar\jmath}\,K_1(\varphi,F,\overline{\varphi})\,F^i\,\partial_+\partial_-\overline{\varphi}^{\bar\jmath} - \frac{4}{M}\partial_i\partial_k\,\partial_{\bar\jmath}\,K_1(\varphi,F,\overline{\varphi})\,\psi_+^i\,\psi_-^k\partial_+\partial_-\overline{\varphi}^{\bar\jmath} + \\
&+\frac{4}{M}\partial_i\,\partial_{\bar\jmath}\,\partial_{\bar\ell}K_1(\varphi,F,\overline{\varphi})\,F^i\,\partial_+\overline{\varphi}^{\bar\jmath}\partial_-\overline{\varphi}^{\bar\ell} - \frac{4}{M}\partial_i\,\partial_k\,\partial_{\bar\jmath}\,\partial_{\bar\ell}\,K_1(\varphi,F,\overline{\varphi})\,\psi_+^i\,\psi_-^k\,\partial_+\overline{\varphi}^{\bar\jmath}\partial_-\overline{\varphi}^{\bar\ell}\,.
\end{aligned} \tag{4.22}$$

Here we have used a similar notation to the one used in the expression of the superpotential and defined

$$K_m(\varphi, F, \overline{\varphi}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi^m \, d\xi \, K\left(\varphi + \frac{\xi}{M} F, \overline{\varphi}\right). \tag{4.23}$$

Here, as in the case of the holomorphic superpotential (4.10), ∂_i, $\partial_{\bar{\jmath}}$ are understood as derivative with respect to φ^i and $\overline{\varphi}^{\bar{\jmath}}$ respectively.

Thus, we have found that the infinite series in [15] for the Kähler potential of the nonanticommutative (2,2) sigma-model can be nicely resummed. Actually, we find that the resulting lagrangian can be divided in two parts. The first one, $\mathcal{L}_0$ is identical, up to total derivatives, to the standard (2,2) lagrangian [18] with the only replacement of the Kähler potential $K(\varphi, \overline{\varphi})$ by the smeared function $K_0(\varphi, F, \overline{\varphi})$. This is exactly the same smearing found in the holomorphic superpotential. On the other hand $\mathcal{L}_1$ contains a number of dimension 3 operators multiplying what one might call the first moment of the smeared Kähler potential, $K_1(\varphi, F, \overline{\varphi})$.

One can easily check that the lagrangian is invariant under the supersymmetry transformations (3.14)-(3.15) generated by $Q_\pm$. Remarkably, $\mathcal{L}_0$ and $\mathcal{L}_1$ are *independently* invariant under the residual supersymmetry.[3] Therefore the $Q_\pm$ supersymmetries do not act irreducibly on the lagrangian obtained using the standard prescription to implement the nonanticommutative deformation. On the other hand under a Kähler transformation $K \to K + f(\varphi) + \overline{f}(\overline{\varphi})$ the lagrangian transforms by a total derivative

$$\mathcal{L} \longrightarrow \mathcal{L} + 4\partial_+ \left[\partial_{\bar{\jmath}} \, \overline{f}(\overline{\varphi}) \, \partial_- \overline{\varphi}^{\bar{\jmath}}\right]. \tag{4.24}$$

Notice that this total derivative only involves antiholomorphic fields and therefore is insensitive to the deformation.

We should point out that the full lagrangian $\mathcal{L}_0 + \mathcal{L}_1$ can be written solely in terms of the function $K_0(\varphi, F, \overline{\varphi})$ due to the identity

$$\frac{1}{M} \partial_i K_m(\varphi, F, \overline{\varphi}) = \frac{\partial}{\partial F^i} K_{m-1}(\varphi, F, \overline{\varphi}), \tag{4.25}$$

at the price, however, of introducing derivatives of the deformed Kähler potential with respect to the auxiliary field. Since all dependence of $K_0(\varphi, F, \overline{\varphi})$ on F^i disappears when $M \to \infty$, only $\mathcal{L}_0$ survives in the anticommutative limit, with $K_0(\varphi, F, \overline{\varphi})$ replaced by the undeformed Kähler potential $K(\varphi, \overline{\varphi})$. In this way the standard Kähler sigma model is retrieved [18].

As advertised in the Introduction, the naïve generalization of Zumino's lagrangian (4.21) is invariant with respect to the residual supersymmetry and also compatible with the deformed superspace structure. This robustness was not expected from the start and it is surprising that after all the dust settles we end up with a well-known lagrangian. However, a number of questions arise. A first one is how the usual Ricci-flat condition for the vanishing of the beta-function [20] changes. Also, the structure of the sigma-model instantons is likely to be modified, for example in the case of the $\mathbb{CP}^1$ model.

[3] Actually, the first and second term in eq. (4.21) are together invariant under $Q_\pm$, as well as the third and fourth terms combined.

5. The classical structure of vacua

In the undeformed case, the analysis of the (traslationally invariant) vacuum structure of the theory begins with the study of the effective potential for the scalar fields $V(\phi)_{\text{eff}}$. Its critical points describe then the possible vacua of the theory. Hermiticity of the original theory guarantees that the potential obtained is positive definite, so that the vanishing of $V(\phi)_{\text{eff}}$ implies the existence of a supersymmetric vacuum. In the cases of interest for our analysis the order parameter is the vacuum expectation value of the auxiliary field F^i.

However, once hermiticity is lost it is difficult to obtain general properties. Let us illustrate the point with a few examples. In the first one we start with a generic superpotential $\mathcal{W}(\Phi)$ and a rather simple Kähler potential

$$K(\Phi,\overline{\Phi}) = \delta_{i\bar{\jmath}}\Phi^i\overline{\Phi}^{\bar{\jmath}}\,. \tag{5.1}$$

This Kähler potential does not receive any deformation so the kinetic terms of all component fields of Φ^i are the standard ones. Hence, the only bosonic terms without space-time derivatives in the lagrangian are

$$\delta_{i\bar{\jmath}}F^i\overline{F}^{\bar{\jmath}} + F^i\partial_i\mathcal{W}_0(\varphi,F) + \overline{F}^{\bar{\jmath}}\partial_{\bar{\jmath}}\overline{\mathcal{W}}(\overline{\varphi})\,, \tag{5.2}$$

with $\mathcal{W}_0(\varphi,F)$ given by eq. (4.11). Varying with respect to $\overline{F}^{\bar{\imath}}$ we obtain

$$F^i + \delta^{i\bar{\jmath}}\,\partial_{\bar{\jmath}}\overline{\mathcal{W}}(\overline{\varphi}) = 0\,, \tag{5.3}$$

and substituting F^i in eq. (5.2), as given by the last equation, we find (cf. [23])

$$V(\varphi,\overline{\varphi})_{\text{eff}} = \partial_{\bar{\jmath}}\,\overline{\mathcal{W}}(\overline{\varphi})\,\delta^{i\bar{\jmath}}\,\partial_i\,\mathcal{W}_0\left(\varphi,-\frac{1}{M}\overline{\partial}\,\overline{\mathcal{W}}\right). \tag{5.4}$$

It is straightforward to check that in the anticommutative limit $M\to\infty$ the standard result is recovered.

It is clear from (5.4) that if there are values of $\langle\overline{\varphi}^{\bar{\jmath}}\rangle$ solving eq. (5.3) for which $\langle F^i\rangle = 0$ then eq. (5.4) receives no deformation and the vacuum of the theory will be the same as for the undeformed theory. However, the effective potential (5.4) for the scalars is not positive definite. In fact it is complex and its real part does not seem to have any positivity property in general. The analysis of the behavior of the theory at the critical point should be carried out using a saddle point analysis. Since we are working in euclidean space, it is not clear what this means for the realization of $\mathcal{N}=1/2$ supersymmetry.

The second example we want to consider here involves a non-trivial Kähler potential which receives a deformation. We have seen that the full lagrangian obtained using the standard prescription to implement the nonanticommutative deformation is not irreducible, and that a simple deformation of Zumino's lagrangian [18] as in eq. (4.21) is also invariant under the $\mathcal{N}=1/2$ supersymmetry transformations (3.14) and (3.15). If we are interested in translationally invariant ground states (or critical points) the additional terms (4.22) do not play any rôle. In this case the changes on the previous analysis to include a non-trivial Kähler potential are straightforward. If we drop in the lagrangian all derivative terms[4] we obtain

$$\mathcal{L} = G_{i\bar{\jmath}}(\varphi, F, \overline{\varphi}) F^i \overline{F}^{\bar{\jmath}} + F^i \partial_i \mathcal{W}_0(\varphi, F) + \overline{F}^{\bar{\jmath}}\, \partial_{\bar{\jmath}} \overline{\mathcal{W}}(\overline{\varphi})\,, \tag{5.5}$$

where, according to (4.21),

$$G_{i\bar{\jmath}}(\varphi, F, \overline{\varphi}) = \partial_i \partial_{\bar{\jmath}} K_0(\varphi, F, \overline{\varphi}) \tag{5.6}$$

is the deformed Kähler metric. Varying with respect to $\overline{F}^{\bar{\jmath}}$ we obtain

$$G_{i\bar{\jmath}}(\varphi, F, \overline{\varphi}) F^i + \partial_{\bar{\jmath}} \overline{\mathcal{W}}(\overline{\varphi}) = 0\,. \tag{5.7}$$

In principle, it would be possible to solve this equation for F^i and substituting in the lagrangian we would obtain the effective scalar potential

$$V(\varphi, \overline{\varphi})_{\text{eff}} = G^{i\bar{\jmath}}\Big(\varphi, F(\varphi, \overline{\varphi}), \overline{\varphi}\Big)\, \partial_i \mathcal{W}_0\Big(\varphi, F(\varphi, \overline{\varphi})\Big) \partial_{\bar{\jmath}} \overline{\mathcal{W}}(\overline{\varphi})\,, \tag{5.8}$$

with $F^i(\varphi)$ solving eq. (5.7). However, this is not a very illuminating expression, since getting solving for F^i in (5.7) is in general complicated. Once again (5.8) is complex. It would be interesting to analyze in more detail the behavior of the scalar theory close to its critical points in some examples. We will come back to this issue in the future.

6. Concluding remarks

We have seen that nonanticommutative two-dimensional sigma models admit a closed form in which the deformation affects the Kähler potential and the superpotential. Physically, this deformation corresponds to a smearing of the target space holomorphic coordinates. According to this, the holomorphic superpotential is obtained by averaging its undeformed value between $\varphi^i - F^i/(2M)$ and $\varphi^i + F^i/(2M)$ as shown in eq. (4.11). It is important to stress that although we have derived this relation in two dimensions a similar expression would hold also for the superpotential of the four-dimensional $\mathcal{N} = 1/2$ Wess-Zumino model. The expressions for the scalar potential found in refs. [10, 23] can be actually retrieved using the identity

$$\frac{F^i}{M} \partial_i \mathcal{W}_0(\varphi, F) = \mathcal{W}\left(\varphi + \frac{F}{2M}\right) - \mathcal{W}\left(\varphi - \frac{F}{2M}\right). \tag{6.1}$$

In the case of the deformation of the kinetic part of the sigma-model action we found that it consists of two parts. The first one is just the usual Kähler action form with the Kähler potential deformed to[5]

$$K_0(\varphi, F, \overline{\varphi}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\, K\left(\varphi + \frac{\xi}{M} F, \overline{\varphi}\right). \tag{6.2}$$

[4] We have not analyzed here "stripped" states [21] although they are very likely to appear in this context. In the following paragraphs we consider only space-time independent vacuum expectation values.

[5] We consider the undeformed Zumino's lagrangian with a kinetic term for the scalar fields of the form $\partial_{\bar{\imath}} K(\varphi, \overline{\varphi}) \partial_+ \partial_- \overline{\varphi}^{\bar{\imath}}$, which differs from the lagrangian appearing in ref. [18] by a total derivative.

Together with this, there is a second term which contains higher dimensional operators suppressed by $1/M$ and with couplings governed by the function

$$K_1(\varphi, F, \overline{\varphi}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi d\xi \, K\left(\varphi + \frac{\xi}{M}F, \overline{\varphi}\right). \tag{6.3}$$

It is quite remarkable that our construction allows us to write the action as a term which corresponds to the usual (2,2) action with a deformed Kähler potential $K_0(\varphi, F, \overline{\varphi})$ together a few higher dimensional terms. When written in the form (4.21) it is obvious that F^i remains a non-propagating field, in spite of its now more complicated couplings. Of course the action $\mathcal{L}_0 + \mathcal{L}_1$ can be written, modulo total derivatives, in the canonical form with a kinetic term for the scalars of the form $\partial_i \partial_{\overline{\jmath}} K_0(\varphi, F, \overline{\varphi})(\partial_+ \varphi^i \partial_- \overline{\varphi}^{\overline{\jmath}} + \partial_- \varphi^i \partial_+ \overline{\varphi}^{\overline{\jmath}})$. However, due to the extra dependence of $K_0(\varphi, F, \overline{\varphi})$ there will be new terms in $\mathcal{L}_1$ containing one derivative of the auxiliary field $\partial_\pm F^i$.

As we pointed out, the (2,2) nonanticommutative sigma model can be obtained from the corresponding $\mathcal{N} = 1/2$ four-dimensional theory. Then from the analysis of ref. [2] it follows that an antichiral ring is preserved.

Hence, in this article we have shown that the $\mathcal{N} = 2$ nonanticommutative sigma-models, in spite of the algebraic complications, can be written as a generalization of Zumino's lagrangian [18]. A number of issues, however, remain to be addressed. A first one concerns the quantum structure of the theory and, in particular, the conditions for the vanishing of the beta-function at one-loop. In the undeformed case the beta-function vanish at one loop provided the target space manifold is Ricci flat [20]. It will be interesting to understand how this condition is changed in the nonanticommutative case where, as we have seen above, the two-dimensional deformation induces a fuzziness in the target manifold.

Since nonanticommutative theories are naturally defined in euclidean space, a second question that can be addressed is about the analog of two-dimensional instantons in the deformed sigma model. Actually, concerning the euclidean character of the theory it would be interesting to see whether there is any way to overcome the constraints of ref. [8] to define models in lorentzian space-time (see [22] for some analysis in this direction). This is specially interesting in order to see if the centrally extended superalgebra (3.7) can play any rôle in lorentzian Quantum Field Theory.

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A. Fourier transforms with Grassmann variables

In this appendix we summarize some basic results of the theory of Fourier transforms for anticommuting variables. Given a function $f(\theta)$ depending on $2n$ anticommuting variables

$\theta^1, \ldots, \theta^{2n}$, its Fourier transform is defined as

$$\widetilde{f}(\eta) = \int d^{2n}\theta \, e^{\eta_a \theta^a} f(\theta) \,, \tag{A.1}$$

where the phase of the integration measure is fixed by requiring

$$\int d^{2n}\theta \, \theta^1 \cdots \theta^{2n} = 1 \,. \tag{A.2}$$

Using that $\delta^{(2n)}(\theta) = \theta^1 \cdots \theta^{2n}$ it is easily proved that the inverse Fourier transform is given by

$$f(\theta) = (-1)^n \int d^{2n}\eta \, e^{-\eta_a \theta^a} \, \widetilde{f}(\eta) \,, \tag{A.3}$$

and the delta function can be represented as

$$\delta^{(2n)}(\theta) = (-1)^n \int d^{2n}\eta \, e^{\eta_a \theta^a} \,. \tag{A.4}$$

Let us analyze now the case of two-dimensional superspace with coordinates $y^{\pm}$, $\theta^{\pm}$. The definition of the Fourier transform simplifies to

$$\widetilde{f}(\eta_{\pm}) = \int d^2\theta \, e^{\eta_+ \theta^+ + \eta_- \theta^-} f(\theta^{\pm}) \,, \tag{A.5}$$

while the inversion formula is

$$f(\theta) = -\int d^2\eta \, e^{-(\eta_+ \theta^+ + \eta_- \theta^-)} \widetilde{f}(\eta_{\pm}) \,. \tag{A.6}$$

Given a general function in superspace it can be decomposed as

$$f(y^{\pm}, \theta^{\pm}) = f_0(y^{\pm}) + \theta^+ f_+(y^{\pm}) + \theta^- f_-(y^{\pm}) + \theta^+ \theta^- f_1(y^{\pm}) \,. \tag{A.7}$$

By applying now the definition (A.1) one finds that its Fourier transform $\widetilde{f}(y^{\pm}, \eta_{\pm})$ with respect to the anticommuting coordinates is given by

$$\widetilde{f}(y^{\pm}, \eta_{\pm}) = f_1(y^{\pm}) + \eta_+ f_-(y^{\pm}) - \eta_- f_+(y^{\pm}) - \eta_+ \eta_- f_0(y^{\pm}) \,. \tag{A.8}$$

That is, upon Fourier transformation with respect to the fermionic coordinates the components of a (2,2) superfield reshuffle. In particular, the highest and lowest components interchange, which implies the useful identities

$$\begin{aligned} f(y^{\pm}, \theta^{\pm}) \Big|_{\theta^+ \theta^-} &= \widetilde{f}(y^{\pm}, \eta_{\pm}) \Big|_{\eta_{\pm} = 0} \,, \\ f(y^{\pm}, \theta^{\pm}) \Big|_{\theta^{\pm} = 0} &= -\widetilde{f}(y^{\pm}, \eta_{\pm}) \Big|_{\eta_+ \eta_-} \,. \end{aligned} \tag{A.9}$$

B. Some details of the calculation of the deformed lagrangian

Here we outline part of the calculations of section 4 leading to the lagrangian of the deformed (2,2) sigma-model. In order to compute the relevant fermionic integrals we are going to forget about the target space indices and use the simplified notation

$$\varphi^{(k)} \equiv \varphi^{i_k}\,, \qquad \psi_\pm^{(k)} \equiv \psi_\pm^{i_k}\,, \qquad F^{(k)} \equiv F^{i_k}\,. \tag{B.1}$$

Therefore the Fourier transform of the chiral superfield $\tilde{\Phi}^{i_k}$ can be written as

$$\begin{aligned}\widetilde{\Phi}^{i_k} \equiv \widetilde{\Phi}^{(k)} &= F^{(k)} + \eta_+\psi_-^{(k)} - \eta_-\psi_+^{(k)} - \eta_+\eta_-\varphi^{(k)} \\ &= F^{(k)}\left[1 - \eta_+\eta_-\frac{\varphi^{(k)}}{F^{(k)}} + \eta_+\frac{\psi_-^{(k)}}{F^{(k)}} - \eta_-\frac{\psi_+^{(k)}}{F^{(k)}}\right] \\ &\equiv F^{(k)}\left[1 - \eta_+\eta_-\hat{\varphi}^{(k)} + \eta_+\hat{\psi}_-^{(k)} - \eta_-\hat{\psi}_+^{(k)}\right],\end{aligned} \tag{B.2}$$

where we have used the notation

$$\hat{\varphi}^{(k)} \equiv \frac{\varphi^{(k)}}{F^{(k)}}\,, \qquad \hat{\psi}_\pm^{(k)} \equiv \frac{\psi_\pm^{(k)}}{F^{(k)}}\,. \tag{B.3}$$

It is important not to confuse this notation with the one for the Weyl symbols introduced in section 2. Now eq. (B.2) can be written as

$$\widetilde{\Phi}^{(k)} = F^{(k)} e^{-\eta_+\eta_-\hat{\varphi}^{(k)}}\left[1 + \eta_+\hat{\psi}_-^{(k)} - \eta_-\hat{\psi}_+^{(k)}\right]. \tag{B.4}$$

Hence, the integral in eq. (4.7) can be expressed as

$$\begin{aligned}(-1)^{n-1}F^{(1)}\cdots F^{(n)}\Bigg[&\int d^2\eta_1\cdots\int d^2\eta_{n-1} e^{\boldsymbol{\eta}_+^T\mathcal{D}\boldsymbol{\eta}_-} + \\ &+\sum_{i<j}^{n}\int d^2\eta_1\cdots\int d^2\eta_{n-1}\Psi^{(i)}\Psi^{(j)}\, e^{\boldsymbol{\eta}_+^T\mathcal{D}\boldsymbol{\eta}_-}\Bigg].\end{aligned} \tag{B.5}$$

Here we have used the vector notation $(\boldsymbol{\eta}_\pm)_i = \eta_{i,\pm}$,

$$\begin{aligned}\Psi^{(k)} &= \eta_{k,+}\psi_+^{(k)} - \eta_{k,-}\psi_-^{(k)}\,, \qquad k = 1,\ldots,n-1\,, \\ \Psi^{(n)} &= \left(-\sum_{i=1}^{n-1}\eta_{i,+}\right)\psi_+^{(n)} - \left(-\sum_{i=1}^{n-1}\eta_{i,-}\right)\psi_-^{(n)}\,,\end{aligned} \tag{B.6}$$

and $\mathcal{D}$ is a $(n-1)\times(n-1)$ matrix given by

$$\mathcal{D} = \begin{pmatrix} -\hat{\varphi}^{(1)} - \hat{\varphi}^{(n)} & \frac{1}{2M} - \hat{\varphi}^{(n)} & \cdots & \frac{1}{2M} - \hat{\varphi}^{(n)} \\ -\frac{1}{2M} - \hat{\varphi}^{(n)} & -\hat{\varphi}^{(2)} - \hat{\varphi}^{(n)} & \cdots & \frac{1}{2M} - \hat{\varphi}^{(n)} \\ \vdots & \vdots & & \vdots \\ -\frac{1}{2M} - \hat{\varphi}^{(n)} & -\frac{1}{2M} - \hat{\varphi}^{(n)} & \cdots & -\hat{\varphi}^{(n-1)} - \hat{\varphi}^{(n)} \end{pmatrix}. \tag{B.7}$$

The first integral within the brackets in eq. (B.5) is equal to

$$\begin{aligned}\det\,\mathcal{D} &= (-1)^{n-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\, P_{n-1}\left(\hat{\varphi}^{(1)}+\frac{\xi}{M},\ldots,\hat{\varphi}^{(n)}+\frac{\xi}{M}\right)\\ &= (-1)^{n-1}\sum_{i=1}^{n}\frac{\partial}{\partial\hat{\varphi}^{(i)}}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\,\left(\hat{\varphi}^{(1)}+\frac{\xi}{M}\right)\cdots\left(\hat{\varphi}^{(n)}+\frac{\xi}{M}\right),\end{aligned} \tag{B.8}$$

where

$$P_k(x_1,\ldots,x_n)=\sum_{i_1<\cdots<i_k}^{n} x_{i_1}\cdots x_{i_k} \tag{B.9}$$

are the elementary symmetric polynomials of degree k. The computation of the second integral in eq. (B.5), that we denote by $\langle\Psi^{(i)}\Psi^{(j)}\rangle$, is a bit more involved. In particular we have to keep in mind that, eventually, all the target space indices of the expression are going to be contracted with the symmetric quantity $\partial_{i_1}\cdots\partial_{i_k}\mathcal{W}(0)$. Therefore in the calculation we only have to retain the symmetric part in the indices and, at the same time, we can simplify expressions by relabeling these indices. After a tedious calculation one arrives at the result

$$\begin{aligned}\sum_{i<j}\langle\Psi^{(i)}\Psi^{(j)}\rangle_S &= (-1)^n\sum_{i<j}^{n}\left[\hat{\psi}_+^{(i)}\hat{\psi}_-^{(j)}-\hat{\psi}_-^{(i)}\hat{\psi}_+^{(j)}\right]\times\\ &\quad\times\frac{\partial^2}{\partial\hat{\varphi}^{(i)}\partial\hat{\varphi}^{(j)}}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi\,\left(\hat{\varphi}^{(1)}+\frac{\xi}{M}\right)\cdots\left(\hat{\varphi}^{(n)}+\frac{\xi}{M}\right),\end{aligned} \tag{B.10}$$

where the subscript S indicates that we have retained only the symmetric part. Plugging now (B.8) and (B.10) back into (B.5) and restoring the indices using (B.4) we arrive at eq. (4.9).

In the case of the antiholomorphic superpotential the calculations leading to (4.17) are simpler than the ones presented above. This is because $\bar{\sigma}^{\bar{\imath}}$ is independent of $\theta^{\pm}$ and then its Fourier transform is proportional to a fermionic delta function. This means that, for example, in a expression like

$$\begin{aligned}(\bar{K}^{\bar{\imath}_1}\star\widetilde{\bar{\sigma}^{\bar{\imath}_2}\star\cdots}\star\bar{\sigma}^{\bar{\imath}_n})(\zeta) &= (-1)^{n-1}\int d^2\eta_1\cdots\int d^2\eta_{n-1}\times\\ &\quad\times\widetilde{\bar{K}}^{\bar{\imath}_1}(\eta_1)\widetilde{\bar{\sigma}}^{\bar{\imath}_2}(\eta_2)\cdots\widetilde{\bar{\sigma}}^{\bar{\imath}_n}\left(\zeta-\sum_{i=1}^{n-1}\eta_i\right)\times\\ &\quad\times\exp\left[\frac{1}{2}\sum_{k<j}^{n-1}\langle\eta_k,\eta_j\rangle\right]\exp\left[-\frac{1}{2}\sum_{i=1}^{n-1}\langle\zeta,\eta_i\rangle\right].\end{aligned} \tag{B.11}$$

Hence, $n-1$ integrations can be readily done and the whole integral is reduced to a single integration. In the case of the term containing $\widetilde{\bar{\lambda}}_+{}^{\bar{\imath}_1}(\eta_1)\widetilde{\bar{\lambda}}_-{}^{\bar{\imath}_2}(\eta_2)$, $n-2$ integrations can be immediately done and one is left with the calculation of two fermionic integrals. Keeping the term proportional to $\zeta_+\zeta_-$ and restricting to the symmetric part in the indices eliminates all dependence on the deformation scale M and the standard undeformed expression is obtained for $\overline{\mathcal{W}}(\overline{\Phi})$.

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Finally we outline the calculation of the deformed Kähler potential. Again the terms $\bar{\sigma}^{\bar{\imath}}$ are mere spectators and the relevant monomials to compute are[6]

$$(\Phi^{i_1}\star\widetilde{\cdots\star\Phi^{i_n}}\star\bar{K}^{\bar{\imath}_1})(\zeta)\Big|_{\zeta_\pm=0}\,,\qquad (\Phi^{i_1}\star\widetilde{\cdots\star\Phi^{i_n}}\star\bar{\lambda}_+^{\bar{\imath}_1}\star\bar{\lambda}_-^{\bar{\imath}_2})(\zeta)\Big|_{\zeta_\pm=0}\,. \tag{B.12}$$

Now we can apply the tricks used to calculate the holomorphic superpotential. As a matter of example we comment on the first kind of monomials where we can identify $\bar{K}^{\bar{\imath}}$ with an extra superfield. This reduces the problem to eq. (B.5) with $n+1$ fields with the formal identifications

$$F^{(n+1)}\equiv 4\partial_+\partial_-\overline{\varphi}^{\bar{\imath}}\,,\qquad \varphi^{(n+1)}\equiv\overline{F}^{\bar{\imath}}\,,\qquad \psi_\pm^{(n+1)}\equiv\mp 2i\partial_\pm\overline{\psi}_+^{\bar{\imath}}\,. \tag{B.13}$$

The calculation now follows the one outlined for the holomorphic superpotential. For example, for the analog of the first term in (B.5) we find

$$\begin{aligned}(-1)^n F^{(1)}\cdots F^{(n)}F^{(n+1)}\int d^2\eta_1\cdots\int d^2\eta_n\, e^{\boldsymbol{\eta}_+^T\mathcal{D}'\boldsymbol{\eta}_-} = \\ =\sum_{i=1}^n F^{(i)}\frac{\partial}{\partial\varphi^{(i)}}\int_{-\frac{1}{2}}^{\frac{1}{2}}d\xi\left(\varphi^{(1)}+\frac{\xi}{M}F^{(1)}\right)\cdots\left(\varphi^{(n)}+\frac{\xi}{M}F^{(n)}\right)\times \\ \times\left(\varphi^{(n+1)}+\frac{\xi}{M}F^{(n+1)}\right)+F^{(n+1)}\int_{-\frac{1}{2}}^{\frac{1}{2}}d\xi\left(\varphi^{(1)}+\frac{\xi}{M}F^{(1)}\right)\cdots\left(\varphi^{(n)}+\frac{\xi}{M}F^{(n)}\right),\end{aligned} \tag{B.14}$$

where by $\mathcal{D}'$ we have denoted the $n\times n$ analog of the matrix (B.7).

We see already here the origin of the higher dimensional terms in eq. (4.22). In the second line of (B.14) we find two terms, one proportional to $\varphi^{(n+1)}=\overline{F}^{\bar{\imath}}$ which will generate the term $F^i\overline{F}^{\bar{\jmath}}$ in (4.21) and a second term proportional to $F^{(n+1)}=4\partial_+\partial_-\overline{\varphi}^{\bar{\imath}}$ which is suppressed by $1/M$ and contains an extra power of ξ. This gives rise to the first term on the right-hand side or (4.22). On the other hand, the last line in eq. (B.14) is at the origin of the first term in eq. (4.21). Remember that eq. (B.14) is globally multiplied by $\overline{\varphi}^{\bar{\imath}_2}\cdots\overline{\varphi}^{\bar{\imath}_n}$.

The remaining terms of the Kähler potential come from the analog of the second term inside the bracket in eq. (B.5) and the ones coming from the second type of monomials in eq. (B.12). Again the calculations can be done mimicking the one for the superpotential.

C. The centrally extended superalgebra

In what follows we are going to discuss in more detail some aspects of the centrally extended supersymmetry algebra

$$Q_\pm^2=P_\pm\,,\qquad \{Q_+,Q_-\}=\frac{1}{M}P_+P_-\,. \tag{C.1}$$

In section 3 we found that, in euclidean space, this type of central extension appears in nonanticommutative deformations (3.1) of (2,2) supersymmetic sigma models. Here,

[6]These monomials multiply $\overline{\varphi}^{\bar{\imath}_2}\cdots\overline{\varphi}^{\bar{\imath}_n}$ and $\overline{\varphi}^{\bar{\imath}_3}\cdots\overline{\varphi}^{\bar{\imath}_n}$ respectively.

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however, we are going to forget about the restriction to euclidean space and play the game of studying the supersymmetry algebra (C.1) in lorentzian signature to discuss its consequence on the spectrum of a theory in Minkowski space in which this superalgebra would be realized.

Unitary representations of the two-dimensional Poincaré group are labelled by the value of the Casimir operator $m \equiv P_+ P_-$. For $m = 0$ the algebra reduces itself to the ordinary (1,1) supersymmetry algebra. Therefore we focus our attention on the representations with $m > 0$. In the rest frame where $P_\pm = m$ and after an obvious rescaling the algebra of supercharges can be written as

$$A^2 = \mathbf{1}\,, \qquad B^2 = \mathbf{1}\,, \qquad \{A, B\} = \xi \mathbf{1}\,, \tag{C.2}$$

where A and B and hermitian operators and $\xi = m/M$.

The study of the unitary representations of the extended algebra (C.2) proceeds by transforming it into an algebra of fermionic creation-annihilation operators. These are defined by

$$a = \alpha A + \beta B\,, \qquad a^\dagger = \alpha^* A + \beta^* B\,. \tag{C.3}$$

The complex constants α, β are fixed by demanding $a^2 = (a^\dagger)^2 = 0$ and $\{a, a^\dagger\} = 1$. Fixing the overall phase ambiguity, one finds the solution

$$\alpha = \frac{e^{i\gamma}}{2\sin(2\gamma)}\,, \qquad \beta = \frac{e^{-i\gamma}}{2\sin(2\gamma)}\,, \tag{C.4}$$

where γ is defined by $\xi = -2\cos(2\gamma)$, which implies the BPS condition

$$|\xi| \leq 2 \quad \Longrightarrow \quad m \leq 2M\,. \tag{C.5}$$

Therefore unitary representations of the algebra exist provided the theory does not contain states with masses above the cutoff $\Lambda = 2M$ given by eq. (C.5). Below this bound unitary irreducible representations contain two states, the fermionic vacuum $|0\rangle$ and $|1\rangle \equiv a^\dagger|0\rangle$.

The previous analysis is valid whenever $|\xi| < 2$. When the BPS bound (C.5) is saturated ($\xi = 2$) the algebra reduces to $A^2 = B^2 = \mathbf{1}$, $\{A, B\} = 2$. Irreducible representations can then be constructed by defining the operators $S_\pm = A \pm B$ which satisfy

$$S_+^2 = 4\,, \qquad S_-^2 = 0\,. \tag{C.6}$$

Since $S_\pm^\dagger = S_\pm$, states which are S_--exact have zero norm and therefore should be removed from the spectrum to preserve unitarity. As a result, irreducible representations are one-dimensional and correspond to the short multiplets of BPS-saturated states.

Here we have studied Hilbert space representations of the deformed algebra (C.1) in two-dimensional Minkowski space-time. A very important question to be answered is, however, whether two-dimensional quantum field theories exist in lorentzian signature in which this algebra of supercharges is realized. Our previous analysis shows that finding such theories would be extremely interesting since they would provide examples of quantum field theories with a built-in cutoff.

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